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Scaling exponents in weakly anisotropic turbulence from the Navier–Stokes equation

By SIEGFRIED GROSSMANN[1], ANNA von der HEYDT[1,2] and DETLEF LOHSE[2]

[1]Fachbereich Physik der Philipps-Universität Marburg, Renthof 6, D-35032 Marburg, Germany

[2]University of Twente, Department of Applied Physics, P.O. Box 217, 7500 AE Enschede, The Netherlands

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The second-order velocity structure tensor of weakly anisotropic strong turbulence is decomposed into its SO(3) invariant amplitudes $d_j(r)$. Their scaling is derived within a scaling approximation of a variable-scale mean-field theory of the Navier–Stokes equation. In the isotropic sector $j = 0$ Kolmogorov scaling $d_0(r) \propto r^{2/3}$ is recovered. The scaling of the higher j amplitudes (j even) depends on the type of the external forcing that maintains the turbulent flow. We consider two options: (i) for an analytic forcing and for decreasing energy input into the sectors with increasing j, the scaling of the higher sectors $j > 0$ can become as steep as $d_j(r) \propto r^{j+2/3}$; (ii) for a non-analytic forcing we obtain $d_j(r) \propto r^{4/3}$ for *all* non-zero and even j.

1. Introduction

In the last few decades scaling in fully developed turbulence has mainly been analysed in terms of the longitudinal velocity structure functions (Monin & Yaglom 1975; Frisch 1995). Meanwhile experimental and numerical evidence has accumulated that, at least for finite Reynolds numbers, the transversal structure functions scale differently (Noullez *et al.* 1997; Grossmann, Lohse & Reeh 1997; Dhruva, Tsuji & Sreenivasan 1997; Chen *et al.* 1997; van de Water & Herweijer 1999). Two questions immediately arise: (i) what is the proper decomposition of the velocity structure tensor into invariant amplitudes, and (ii) what is the origin of their different scalings?

In addressing the first question, Arad, L'vov & Procaccia (1999*b*) suggested decomposing the second-order velocity structure tensor into the amplitudes $d_{jmq}(r)$ of the irreducible SO(3) representation,

$$D_{ik}(\boldsymbol{r}) = \langle\langle v_i(\boldsymbol{r},t)v_k(\boldsymbol{r},t)\rangle\rangle = \sum_{jmq} d_{jmq}(r) B_{ik}^{jmq}(\hat{\boldsymbol{r}}), \tag{1.1}$$

reflecting the rotational symmetry of the Navier–Stokes equation. Here, $v_i(\boldsymbol{r},t) = u_i(\boldsymbol{x}+\boldsymbol{r},t) - u_i(\boldsymbol{x},t)$ is the velocity difference, the brackets $\langle\langle\ldots\rangle\rangle$ denote the ensemble average and, as in Arad *et al.* (1999*b*), the tensors $B_{ik}^{jmq}(\hat{\boldsymbol{r}})$ are combinations of the spherical harmonics $Y_{jm}(\hat{\boldsymbol{r}})$ and operations like ∂_{r_i}, r_k, δ_{ik}, the index q labels the different types of such combinations, and $\hat{\boldsymbol{r}}$ denotes the unit vector in the direction of $\boldsymbol{r}$, $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$. Non-zero values of j contribute to D_{ik} if the turbulence is not isotropic.

In Arad *et al.* (1998), Arad *et al.* (1999*a*), Kurien & Sreenivasan (2000), Kurien *et al.* (2000), and Kurien & Sreenivasan (2001) the scaling exponents of the amplitudes $d_j(r)$ were extracted from experimental as well as numerical data. For $j = 0$ a scaling exponent close to the Kolmogorov value 2/3 was recovered, but for $j = 2$ values close to 4/3 were found. This scaling exponent corresponds to a power spectrum $\sim k^{-7/3}$, a behaviour first suggested for shear flow by Lumley (1967) through a dimensional argument. Experimental evidence for it was found by Wyngaard & Cote (1972) and, later, also by Saddoughi & Veeravalli (1994), among others. For higher $j > 2$ Biferale & Toschi (2001) have found even larger scaling exponents from the analysis of numerical data, namely 1.67–1.7 for $j = 4$ and 3.3–3.4 for $j = 6$.

Here we aim at analytically *calculating* the mean field part of the scaling exponents of the j-amplitudes from the Navier–Stokes equation for weakly anisotropic, homogeneous turbulence. We employ the variable-scale mean-field theory of Effinger & Grossmann (1987), i.e. we disregard intermittency corrections.

2. The Effinger–Grossmann mean field theory for the weakly anisotropic case

The main idea in Effinger & Grossmann (1987) is to decompose the velocity field into a smooth part $u_i^{(r)}$, defined as spatial average over a sphere with variable radius r and therefore containing only scales larger than r, and a strongly varying part $\tilde{u}_i^{(r)}$, to which the scales smaller than r contribute. Within the Effinger–Grossmann theory, not only can the K41 r-scaling exponent 2/3 of the structure function be analytically calculated from the Navier–Stokes equation, but also the Kolmogorov constant $b = 6.3$. As we now assume (weak) anisotropy of the flow, we introduce an average which reflects its scale r and, in addition, the direction of the averaging. Therefore, for each component, we choose an average over a line in the $\hat{\boldsymbol{r}}$-direction with the length $2r$,

$$u_i^{(\boldsymbol{r})}(\boldsymbol{x},t) = \frac{1}{2r}\int_{-r}^{r} u_i(\boldsymbol{x}+y\hat{\boldsymbol{r}},t)\mathrm{d}y \equiv \langle u_i(\boldsymbol{x}+y\hat{\boldsymbol{r}},t)\rangle_y^{(\boldsymbol{r})}. \tag{2.1}$$

Correspondingly, $\tilde{u}_i^{(\boldsymbol{r})}(\boldsymbol{x},t) = u_i(\boldsymbol{x},t) - u_i^{(\boldsymbol{r})}(\boldsymbol{x},t)$. The upper index $\boldsymbol{r}$ denotes that these averages not only depend on the scale r, but also on the direction $\hat{\boldsymbol{r}}$ of averaging, thus on the full vector $\boldsymbol{r}$. The lower index y indicates the averaged variable. As in the original spherical averaging case there is a close relation between the second order moments of $u_i^{(\boldsymbol{r})}$ and the structure tensor $D_{ik}(\boldsymbol{r})$: $\langle\langle u_i^{(\boldsymbol{r})}u_k^{(\boldsymbol{r})}\rangle\rangle = \langle\langle u_iu_k\rangle\rangle - \frac{1}{2}\langle\langle D_{ik}(\boldsymbol{y}_1+\boldsymbol{y}_2)\rangle_{y_1}^{(\boldsymbol{r})}\rangle_{y_2}^{(\boldsymbol{r})}$. This relation is crucial for the method. For simplicity we use the abbreviation $\boldsymbol{y} = y\hat{\boldsymbol{r}}$. In the above double average $\boldsymbol{y}_1$ is thus parallel to $\boldsymbol{y}_2$.

Eliminating the pressure p gives a non-local term involving the Green function $G(x)$. Inserting the velocity decomposition into the Navier–Stokes equation and averaging, we obtain an equation of motion for the large-scale ('superscale') velocity:

$$\begin{aligned}\partial_t u_i^{(\boldsymbol{r})}(\boldsymbol{x},t) = &-u_j^{(\boldsymbol{r})}(\boldsymbol{x},t)\partial_{x_j}u_i^{(\boldsymbol{r})}(\boldsymbol{x},t) - \langle\tilde{u}_j^{(\boldsymbol{r})}(\boldsymbol{x}+y\hat{\boldsymbol{r}})\partial_{x_j}\tilde{u}_i^{(\boldsymbol{r})}(\boldsymbol{x}+y\hat{\boldsymbol{r}})\rangle_y^{(\boldsymbol{r})}\\ &+\nu\Delta_x u_i^{(\boldsymbol{r})}(\boldsymbol{x},t) + f_i^{(\boldsymbol{r})}(\boldsymbol{x},t)\\ &+\int \mathrm{d}^3x' G(x')\partial_{x_i'}\{u_{k|l}^{(\boldsymbol{r})}(\boldsymbol{x}+\boldsymbol{x}',t)u_{l|k}^{(\boldsymbol{r})}(\boldsymbol{x}+\boldsymbol{x}',t)\\ &\qquad+\langle\tilde{u}_{k|l}^{(\boldsymbol{r})}(\boldsymbol{x}+\boldsymbol{x}'+y\hat{\boldsymbol{r}},t)\tilde{u}_{l|k}^{(\boldsymbol{r})}(\boldsymbol{x}+\boldsymbol{x}'+y\hat{\boldsymbol{r}},t)\rangle_y^{(\boldsymbol{r})}\}.\end{aligned} \tag{2.2}$$

We use the abbreviation $u_{i|k}(\boldsymbol{x},t) := \partial_{x_k}u_i(\boldsymbol{x},t)$, etc.; Δ_x denotes the Laplacian with respect to $\boldsymbol{x}$, ν is the kinematic viscosity, and f_i an external forcing maintaining the turbulent flow. In Effinger & Grossmann (1987) isotropic forcing is considered. Here, by proper choice of f_i we explicitly introduce anisotropy. It implies an anisotropic energy input whose characteristic details will be discussed later. Subtracting (2.2) from the Navier–Stokes equation gives an equation for the 'subscale' velocity $\tilde{u}_i^{(r)}$. Its formal solution can be found by time integrating along a Lagrangian path $\boldsymbol{x}(t';\boldsymbol{z},t)$ of a fluid particle which at time $t'=t$ is at the position $\boldsymbol{x}=\boldsymbol{z}$. Also, from equation (2.2) we can derive an energy balance equation for the superscales.

The central approximation of the mean-field theory of Effinger & Grossmann (1987) is that the small-scale flow is statistically independent of the smooth large-scale one. Therefore, in higher-order moments we factorize the $u^{(r)}$ from the $\tilde{u}^{(r)}$, e.g.

$$\langle\langle \tilde{u}^{(r)}\tilde{u}^{(r)}u^{(r)}u^{(r)}\rangle\rangle \simeq \langle\langle \tilde{u}^{(r)}\tilde{u}^{(r)}\rangle\rangle\langle\langle u^{(r)}u^{(r)}\rangle\rangle.$$

Physically this means that the large scales feel the small ones as a kind of eddy viscosity. Note again that this factorization excludes intermittency effects. Another assumption is that in the time integration along a Lagrangian path of a fluid particle the slow t'-dependence of the superscales $u^{(r)}$ is neglected since the subscales $\tilde{u}^{(r)}$ fluctuate on a much shorter time scale.

The resulting contributions to the energy balance can be expressed in terms of the structure function tensor $D_{ik}(\boldsymbol{r})$. To simplify the expressions we introduce the second-order moment of the superscale velocity, $R_{ik}^{(r)}(\boldsymbol{r}')$, and the time-integrated correlation function of the subscale eddies, $N_{ik}^{(r)}(\boldsymbol{r}')$. Both can be expressed in terms of the structure function tensor:

$$\begin{aligned} R_{ik}^{(r)}(\boldsymbol{r}') &:= \langle\langle\langle u_i^{(r)}(\boldsymbol{x},t)u_k^{(r)}(\boldsymbol{x}+\boldsymbol{y}+\boldsymbol{r}',t)\rangle\rangle\rangle_{y}^{(r)} \\ &= \langle\langle u_i u_k\rangle\rangle - \tfrac{1}{2}\langle\langle\langle D_{ik}(\boldsymbol{r}'+\boldsymbol{y}_1+\boldsymbol{y}_2+\boldsymbol{y}_3)\rangle_{y_1}^{(r)}\rangle_{y_2}^{(r)}\rangle_{y_3}^{(r)}, \\ N_{ik}^{(r)}(\boldsymbol{r}') &:= \int_{-\infty}^{t} \mathrm{d}t'\langle\langle \tilde{u}_i^{(r)}(\boldsymbol{z},t)\tilde{u}_k^{(r)}(\boldsymbol{x}(t';\boldsymbol{z},t)+\boldsymbol{r}',t')\rangle\rangle. \end{aligned}$$

$N_{ik}^{(r)}$ probes the (Lagrangian) dynamics and can be considered as an eddy transport coefficient for the superscale flow. To obtain a closed set of equations we express $N_{ik}^{(r)}$ in terms of equal time and therefore stationary *static* objects like the structure tensor $D_{ik}(\boldsymbol{r})$. This is achieved by continued fraction projector expansion (Grossmann & Thomae 1982; Daems *et al.* 1999). With the static subscale correlation $\tilde{C}_{ik}^{(r)}(\boldsymbol{x}') := \langle\langle \tilde{u}_i^{(r)}(\boldsymbol{x},t)\tilde{u}_k^{(r)}(\boldsymbol{x}+\boldsymbol{x}',t)\rangle\rangle$ and the frequency matrix $\tilde{\varGamma}_{ik}^{(r)}(\boldsymbol{x}') := -\langle\langle \tilde{u}_i^{(r)}(\boldsymbol{z},t)d_{t'}\tilde{u}_k^{(r)}(\boldsymbol{x}(t';\boldsymbol{z},t)+\boldsymbol{x}',t')\rangle\rangle|_{t'=t}$ we can write $N_{ik}^{(r)}$ in a 1-pole approximation as

$$N_{ik}^{(r)}(\boldsymbol{x}') = \tilde{C}_{ij}^{(r)}(\boldsymbol{x}')(\tilde{\varGamma}^{(r)}(\boldsymbol{x}'))_{jl}^{-1}\tilde{C}_{lk}^{(r)}(\boldsymbol{x}').$$

The tensor $\tilde{\boldsymbol{C}}$ and the frequency matrix $\tilde{\boldsymbol{\varGamma}}$ can be expressed in terms of the structure function tensor

$$\begin{aligned} \tilde{C}_{ik}^{(r)}(\boldsymbol{x}') &= -\tfrac{1}{2}\langle\langle D_{ik}(\boldsymbol{x}'+\boldsymbol{y}_1+\boldsymbol{y}_2)\rangle_{y_1}^{(r)}\rangle_{y_2}^{(r)} + \langle D_{ik}(\boldsymbol{x}'+\boldsymbol{y})\rangle_{y}^{(r)} - \tfrac{1}{2}D_{ik}(\boldsymbol{x}'), \\ \tilde{\varGamma}_{ik}^{(r)}(\boldsymbol{x}') &= \tfrac{2}{3}\epsilon\delta_{ik} - 2\nu\langle\Delta_y D_{ik}(\boldsymbol{y})\rangle_{y}^{(r)} \\ &\quad +\nu\langle\langle\Delta_{y_1}D_{ik}(\boldsymbol{y}_1+\boldsymbol{y}_2)\rangle_{y_1}^{(r)}\rangle_{y_2}^{(r)} + \nu\Delta_{x'}\tilde{C}_{ik}^{(r)}(\boldsymbol{x}'). \end{aligned}$$

For more details compare with the case of isotropic turbulence in Effinger & Grossmann (1987). In a general anisotropic case the dissipation matrix elements $\nu\langle\langle u_{i|j}^2\rangle\rangle$ might be different for different i,j. Here, in the weakly anisotropic case, we assume that the anisotropy corrections are small on the scales where dissipation takes place. Therefore we insert one total dissipation rate per unit mass $\epsilon = \nu\langle\langle u_{i|j}u_{i|j}\rangle\rangle$ (summation implied).

The superscale energy balance equation contains three contributions for the losses. $E_d(\boldsymbol{r})$ describes the direct viscous energy dissipation by the superscale eddies. The other two, consisting of a local and non-local part of $E_t(\boldsymbol{r})$, account for the energy transfer from the large to the small scales. These losses are balanced by the energy input rate $E_{in}(\boldsymbol{r})$ caused by the external forcing:

$$E_d(\boldsymbol{r}) + E_{t,lo}(\boldsymbol{r}) + E_{t,nolo}(\boldsymbol{r}) = E_{in}(\boldsymbol{r}). \tag{2.3}$$

As in Effinger & Grossmann (1987) the three different contributions can be written

$$E_d(\boldsymbol{r}) = \tfrac{1}{2}\nu\langle\langle\Delta_{y_1} D_{ii}(\boldsymbol{y}_1 + \boldsymbol{y}_2)\rangle_{y_1}^{(r)}\rangle_{y_2}^{(r)}, \tag{2.4}$$

$$E_{t,lo}(\boldsymbol{r}) = -\tfrac{1}{2} N_{jk}^{(r)}(\boldsymbol{x}' = 0)\partial_{x_j'}\partial_{x_k'} R_{ii}^{(r)}(\boldsymbol{x}')|_{\boldsymbol{x}'=0}, \tag{2.5}$$

$$\begin{aligned} E_{t,nolo}(\boldsymbol{r}) = &-\int \mathrm{d}^3x' G(x')\partial_{x_i'}\partial_{x_j'}\{\partial_{x_j'} N_{lk}^{(r)}(\boldsymbol{x}') - \partial_{x_l'} N_{jk}^{(r)}(\boldsymbol{x}')\}\partial_{x_k'} R_{il}^{(r)}(\boldsymbol{x}') \\ &+ \int \mathrm{d}^3x' G(x')\Delta_{x'}(\partial_{x_i'} N_{lk}^{(r)}(\boldsymbol{x}'))\partial_{x_k'} R_{il}^{(r)}(\boldsymbol{x}'). \end{aligned} \tag{2.6}$$

The energy input rate is given by

$$E_{in}(\boldsymbol{r}) = \langle\langle u_i^{(r)} f_i^{(r)}\rangle\rangle. \tag{2.7}$$

Note that in contrast to the isotropic case all terms in the energy balance (2.3) now depend on the vector $\boldsymbol{r}$, not merely on its absolute value, the scale r.

Equation (2.3) together with (2.4)–(2.7) constitute a set of integro -differential equations for the tensor $D_{ik}(\boldsymbol{r})$. Now, anisotropy is assumed to be small. More precisely, in a SO(3)-decomposition of $D_{ik}(\boldsymbol{r})$ the j-amplitudes are assumed to decrease in magnitude for higher angular wavenumber j. Then (2.3) can be solved order by order in j, to give the structure function amplitudes $d_j(r)$. They will not be universal but depend on the anisotropy of the forcing. However, what we may hope is that the *scaling* of the individual j-amplitudes *is universal.* To analyse this, it is sufficient to focus on the scaling behaviour of the various contributions in (2.3).

Scalewise, multiple spatial averages can be reduced to first-order ones, e.g. $E_d(\boldsymbol{r}) \sim \nu/2\langle\Delta_y D_{ii}(\boldsymbol{y})\rangle_y^{(r)}$, and local and non-local energy transfer rates scale with the same exponent. Here and in the following $\sim$ has the meaning of 'scalewise equal'. Thus the energy balance equation scalewise simplifies to

$$E_{in}(\boldsymbol{r}) \sim \left\langle\frac{\nu}{2}\Delta_y D_{ii}(\boldsymbol{y})\right\rangle_y^{(r)} + \left\langle\frac{\alpha}{\epsilon} D_{jl}(\boldsymbol{r}) D_{lk}(\boldsymbol{r})\partial_{y_j}\partial_{y_k} D_{ii}(\boldsymbol{y})\right\rangle_y^{(r)}.$$

Here, ϵ is the mean energy dissipation rate per unit mass, and the constant α takes into account the relative weight of the transport terms, E_t in (2.3). Scalewise this equation can be simplified even further:

$$E_{in}(\boldsymbol{r}) \sim \frac{1}{2}\left(\nu + \frac{\beta}{\epsilon}\boldsymbol{D}(\boldsymbol{r})\boldsymbol{D}(\boldsymbol{r})\right)\Delta\boldsymbol{D}(\boldsymbol{r}), \tag{2.8}$$

where β takes into account the missing constants of proportionality. $\boldsymbol{DD}$ and $\Delta\boldsymbol{D}$ stand for the tensorial products of two structure function tensors and of a second-order spatial derivative of the structure function tensor, respectively.

3. SO(3)-decomposition

Taking into account the full tensorial character of $D_{ik}(\boldsymbol{r})$ (equation (1.1)) complicates the resulting equation. Therefore, for simplicity, we assume that the r-scaling behaviour remains the same. As we are interested at present in the scaling exponents only, we disregard the tensorial character of the structure function (i.e. drop the index q of $d_{jmq}(r)$) and expand into spherical harmonics:

$$D(\boldsymbol{r}) \simeq d_{00}(r)Y_{00} + \sum_m d_{2m}(r)Y_{2m}(\hat{\boldsymbol{r}}) + \sum_m d_{4m}(r)Y_{4m}(\hat{\boldsymbol{r}}) + \dots \\ \sim \sum_j d_j(r) \sum_m Y_{jm}(\hat{\boldsymbol{r}}). \tag{3.1}$$

Here, we assume that the scaling behaviour of $d_{jm}(r)$ is – for fixed j – the same for all m, and therefore simply write $d_j(r)$. We analogously expand the energy input rate into spherical harmonics:

$$E_{in}(\boldsymbol{r}) = \sum_{j,m} e_{jm}(r)Y_{jm}(\hat{\boldsymbol{r}}) \sim \sum_j e_j(r) \sum_m Y_{jm}(\hat{\boldsymbol{r}}), \tag{3.2}$$

where

$$e_{jm}(r) = \int \mathrm{d}(\cos\theta)\mathrm{d}\varphi \;\; Y^*_{jm}(\hat{\boldsymbol{r}})E_{in}(\boldsymbol{r}). \tag{3.3}$$

Then we insert the SO(3)-decomposition (3.1) of the structure function and the corresponding expansion (3.2) of the energy input rate into equation (2.8).

From now on we only focus on the inertial subrange (ISR), $\eta \ll r \ll L$, where η is the Kolmogorov length, in which the second term on the right-hand side of equation (2.8) dominates. Thus the energy balance equation is

$$\sum_j e_j(r) \sum_m Y_{jm}(\hat{\boldsymbol{r}}) \sim \frac{\beta}{r^2} \left(\sum_j d_j(r) \sum_m Y_{jm}(\hat{\boldsymbol{r}}) \right)^3. \tag{3.4}$$

Projecting equation (3.4) on the different j-sectors and taking into account only the first three j ($j = 0, 2, 4$) yields three nonlinear equations for $d_0(r)$, $d_2(r)$ and $d_4(r)$:

$$e_0(r)r^2 \sim \beta[A_{000,0}(d_0(r))^3 + 3A_{022,0}d_0(r)(d_2(r))^2 + 3A_{044,0}d_0(r)(d_4(r))^2 + A_{222,0}(d_2(r))^3 \\ +3A_{224,0}(d_2(r))^2d_4(r) + 3A_{244,0}d_2(r)(d_4(r))^2 + A_{444,0}(d_4(r))^3], \tag{3.5a}$$

$$e_2(r)r^2 \sim \beta[3A_{002,2}(d_0(r))^2d_2(r) + 3A_{022,2}d_0(r)(d_2(r))^2 + 3A_{044,2}d_0(r)(d_4(r))^2 \\ +6A_{024,2}d_0(r)d_2(r)d_4(r) + A_{222,2}(d_2(r))^3 + 3A_{224,2}(d_2(r))^2d_4(r) \\ +3A_{244,2}d_2(r)(d_4(r))^2 + A_{444,2}(d_4(r))^3], \tag{3.5b}$$

$$e_4(r)r^2 \sim \beta[3A_{004,4}(d_0(r))^2d_4(r) + 3A_{022,4}d_0(r)(d_2(r))^2 + 3A_{044,4}d_0(r)(d_4(r))^2 \\ +6A_{024,4}d_0(r)d_2(r)d_4(r) + A_{222,4}(d_2(r))^3 + 3A_{224,4}(d_2(r))^2d_4(r) \\ +3A_{244,4}d_2(r)(d_4(r))^2 + A_{444,2}(d_4(r))^3]. \tag{3.5c}$$

Here, $A_{j_1j_2j_3,j_4} = \sum_{m_1,m_2,m_3,m_4} \int \mathrm{d}(\cos\theta)\mathrm{d}\varphi Y^*_{j_4m_4} Y_{j_1m_1} Y_{j_2m_2} Y_{j_3m_3}$. The $A_{j_1j_2j_3,j_4}$ can have either sign.

To extract the scaling laws for the different $d_j(r)$, equations (3.5) have to be solved. But before doing so, we have to specify the energy input rate $E_{in}(\boldsymbol{r})$, equation (2.7), which depends on the external forcing $f_i^{(r)}$.

4. Anisotropic forcing

In the isotropic and homogeneous case $E_{in}(\boldsymbol{r}) = E_{in}$ is a scale-independent constant (Effinger & Grossmann (1987)). The reason is the following. While the superscale

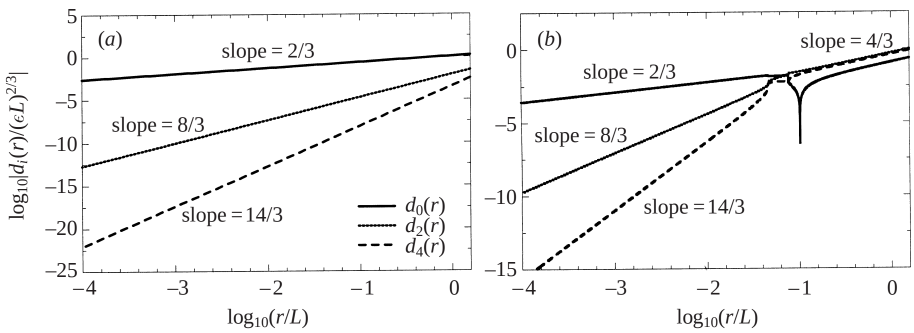

FIGURE 1. Scaling behaviour of the amplitudes of the second-order structure function $d_0(r)$, $d_2(r)$ and $d_4(r)$, for an analytic forcing. (*a*) Strong isotropic forcing together with weak anisotropy corrections, $e_0/\epsilon = 0.89$, $e_2/\epsilon = 0.1$, $e_4/\epsilon = 0.01$. (*b*) Strong anisotropic forcing: the first anisotropic sector $j = 2$ dominates the energy input, $e_0/\epsilon = 0.001$, $e_2/\epsilon = 0.989$, $e_4/\epsilon = 0.01$. This might already reach the limits of our assumptions regarding *weak* anisotropy. The dip of the d_0-curve originates from a change of sign of $d_0(r)$.

velocity field $u_i^{(r)}$ contains all scales larger than r, the forcing $f_i^{(r)}$ has the outer scale L only. For each $r \leqslant L$ the complete forcing is included in the same and therefore r-independent way. Of course, $E_{in} = \epsilon$. In the present case, however, the forcing has to provide an anisotropic flow. As a consequence we shall find that $f_i^{(r)}$ has to depend on *all* scales r, implying that the energy input rate will also depend on all r.

We will discuss two different classes of anisotropic flows: a general analytic forcing and a non-analytic forcing. For both we can determine the scaling behaviour with dimensional arguments.

4.1. *Analytic forcing*

Let us assume that the forcing $f_i(\boldsymbol{x}) \sim a_i \boldsymbol{k} \cdot \boldsymbol{r} \sin(\boldsymbol{k} \cdot \boldsymbol{x})$ and the velocity profile $u_i(\boldsymbol{x}) \sim b_i \boldsymbol{k} \cdot \boldsymbol{r} \sin(\boldsymbol{k} \cdot \boldsymbol{x})$ depend on one input wavenumber $\boldsymbol{k}$ only. They are analytic in the components of position $\boldsymbol{x}$ and the scale vector $\boldsymbol{r}$. To fulfil the incompressibility condition, $\partial_i f_i = 0$ and $\partial_i u_i = 0$, the vectors a_i and b_i must be chosen as $a_i k_i = b_i k_i = 0$. Then, applying the y-average defined in equation (2.1) yields $u_i^{(r)} \sim f_i^{(r)} \sim [\cos(\boldsymbol{k} \cdot (\boldsymbol{x} + \boldsymbol{r})) - \cos(\boldsymbol{k} \cdot (\boldsymbol{x} - \boldsymbol{r}))]$. Therefore

$$E_{in}(\boldsymbol{r}) \sim \langle\!\langle (\cos(\boldsymbol{k} \cdot (\boldsymbol{x} + \boldsymbol{r})) - \cos(\boldsymbol{k} \cdot (\boldsymbol{x} - \boldsymbol{r})))^2 \rangle\!\rangle = 1 - \cos(2kr\xi) \tag{4.1}$$

with $\xi = \cos\theta$, the projection on the z-axis defined by $\hat{\boldsymbol{k}}$. A power series expansion of $E_{in}(\boldsymbol{r})$ in the variable $r\xi$ inserted into equation (3.3) implies (because $\xi^n \perp Y_{jm}$ for all $n < j$) that $e_{jm} \sim r^j$ plus higher powers.

We now solve equations (3.5) and extract the power laws for the different $d_j(r)$. Figure 1 shows the solutions of (3.5). In (*a*) the isotropic part of the energy input e_0 is the largest one, and the anisotropy contributions are small corrections. In this case, over the whole calculated range $10^{-4} \leqslant r/L \leqslant 1$ the $d_j(r)$ scale as

$$d_j(r) \sim r^{j+2/3}. \tag{4.2}$$

We can see this scaling behaviour easily from equations (3.5): since $d_4 \ll d_2 \ll d_0$, the dominating term on the right hand side of equation (3.5*a*) is $A_{000,0}(d_0)^3$. It is balanced by $e_0 r^2$. Therefore, $d_0 \sim r^{2/3}$. Then, in equations (3.5*b*, *c*) the leading terms $A_{002,2}(d_0)^2 d_2$ and $A_{004,4}(d_0)^2 d_4$ are balanced by $e_2 r^2 \sim r^4$ and $e_4 r^2 \sim r^6$, respectively.

Therefore, we expect $d_2 \sim r^{8/3}$ and $d_4 \sim r^{14/3}$. Though for $j = 0$ we recover the mean-field scaling of the isotropic amplitude of the structure function $d_0 \sim r^{2/3}$, as in Effinger & Grossmann (1987), the result for the $j = 2$ sector is at variance with the experimental finding by Kurien *et al.* (2000), who found a scaling exponent close to 4/3. If, on the other hand, we chose a strongly anisotropic energy input with $e_2 \gg e_0, e_4$ (in which case the assumption of weak anisotropy of course breaks down), then the r-scaling range with $d_j(r) \sim r^{j+2/3}$ becomes smaller, while at larger values of r a new scaling range $d_j \sim r^{4/3}$ with the same exponent 4/3 for all j emerges, see figure 1(*b*). For $j = 2$ this finding is now consistent with the experimental observations by Kurien *et al.* (2000). However, it is inconsistent with the exponent 2/3 to be expected for the $j = 0$ amplitude. In summary, the analytic energy input does not seem to describe the experimental findings. We therefore now explore the option of non-analytic forcing.

4.2. *Non-analytic forcing*

We consider a shear flow with its shear in the 3-direction. Then the three f-components are different. We decompose the velocity u_i and the forcing f_i into an isotropic (*iso*) and a (smaller) anisotropic (*an*) part: $u_i = u_i^{(iso)} + u_i^{(an)}$, $f_i = f_i^{(iso)} + f_i^{(an)}$. Then, at first order of anisotropy

$$\langle\langle u_i^{(\boldsymbol{r})} f_i^{(\boldsymbol{r})} \rangle\rangle \simeq \langle\langle u_i^{(\boldsymbol{r})} f_i^{(\boldsymbol{r})(iso)} \rangle\rangle + \langle\langle u_i^{(\boldsymbol{r})(iso)} f_i^{(\boldsymbol{r})(an)} \rangle\rangle = E_{in}^{(iso)} + E_{in}^{(an)}(\boldsymbol{r}). \tag{4.3}$$

Repeating the arguments at the beginning of §4 for the isotropic case the first term on the right-hand side does not depend on r, i.e. $E_{in}^{(iso)} \sim r^0$. Namely, since $f_i^{(\boldsymbol{r})(iso)}$ has scales of order L only, the smaller scales in the products with $u_i^{(\boldsymbol{r})(iso)}$ or $u_i^{(\boldsymbol{r})(an)}$ cannot contribute, irrespective of their degree of isotropy. The second term, however, will depend on r and introduces anisotropy.

Let us determine $E_{in}^{(an)}(\boldsymbol{r})$ by scaling arguments. The flow profile in shear flow is generated by the boundary conditions: one plate is moving with velocity U, the other one is at rest. These boundary conditions have to be mimicked by the forcing f in an infinitely extending flow. The linear mean velocity profile $(U/L)z$ (and therefore also the corresponding f) has Fourier coefficients on all scales,

$$u^{(an)}(k) = \frac{U}{L^2}\int_{-L}^{L} \mathrm{d}z\ z\mathrm{e}^{\mathrm{i}kz} = 2\mathrm{i}U\left(\frac{\sin kL}{k^2L^2} - \frac{\cos kL}{kL}\right).$$

In the case of large k, i.e. $k^{-1} \sim z \ll L$, the second term dominates. We therefore asymptotically find

$$u^{(an)}(k) \sim \frac{\cos kL}{kL} \sim \frac{1}{k} \sim z = r\cos\theta. \tag{4.4}$$

Incidentally, a parabolic velocity profile as in pipe flow, $(U/L^2)z^2$, gives the same asymptotic scaling, $u^{(an)}(k) \sim (\sin kL)/kL \sim 1/k \sim z$ for large k.

Next, we determine the r-dependence of $f^{(an)}$. From the Navier–Stokes equation we have $\partial u/\partial t = \cdots + f$. Therefore, the dimension and r-scaling of f must correspond to that of u/τ, where τ is the r-eddy turnover time. In the isotropic case the turnover time τ scales like $\tau(r) \sim r/u(r) \sim r/r^{1/3} \sim r^{2/3}$. We use the r-dependence of the anisotropic velocity field $u^{(an)}(r)$ together with that of the isotropic turnover time $\tau(r)$ to estimate the scaling of the anisotropic forcing $f^{(an)}$ in first order. Since $u^{(an)}$ behaves as $u^{(an)} \sim r\cos\theta$ according to equation (4.4), we have $f^{(an)} \sim r(\cos\theta)/r^{2/3} \sim r^{1/3}\cos\theta$. Note that this anisotropic forcing scales as the isotropic velocity $u^{(iso)} \sim r^{1/3}$. Then

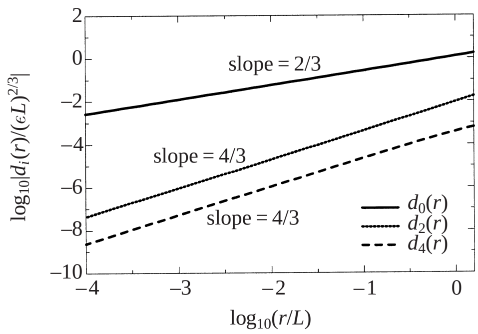

FIGURE 2. Scale dependence of the amplitudes $d_0(r)$, $d_2(r)$ and $d_4(r)$ of the second-order structure function, for a non-analytic forcing. The forcing is assumed to be predominantly isotropic with small anisotropy corrections, $e_0/\epsilon = 0.89$, $e_2/\epsilon = 0.1$, $e_4/\epsilon = 0.01$.

both factors of $E_{in}^{(an)}(\boldsymbol{r})$ in (4.3) not only contain all scales, but also the same power-law exponents. We therefore find

$$E_{in}^{(an)}(\boldsymbol{r}) \sim r^{2/3} \cos\theta. \tag{4.5}$$

The forcing still has an additional factor $g(\theta, \varphi)$, which can be chosen such that the incompressibility constraint $\partial_i f_i = 0$ is fulfilled. Knowing now the scaling behaviour of the energy input rate, we proceed to expand it into spherical harmonics (see equation (3.2)),

$$E_{in}(\boldsymbol{r}) = E_{in}^{(iso)} + r^{2/3}\tilde{E}_{in}^{(an)}(\hat{\boldsymbol{r}}) = \sum_{j,m} e_{jm}(r) Y_{jm}(\hat{\boldsymbol{r}}).$$

Here, $e_{00}(r)$ equals $E_{in}^{(iso)}$, while for $j > 0$ the input rate amplitudes are $e_{jm}(r) = \int \mathrm{d}(\cos\theta)\mathrm{d}\varphi\ Y_{jm}^{*}(\hat{\boldsymbol{r}})E_{in}^{(an)}(\boldsymbol{r})$. Here the r- and ξ-dependences are not coupled as $r\xi$, in contrast to the analytic case treated above. Thus there is no j-dependence of the leading r-power of the input amplitudes e_{jm}. The lowest j-value projection $e_{jm}(r)$ of the anisotropy correction, in general $j = 2$, and all the higher ones, have the same power law, here according to (4.5) $\sim r^{2/3}$. The physics behind this decoupling of the r- and ξ-dependence is that a shear profile – in contrast to a single input wavenumber $\boldsymbol{k}$ – contains all wavenumbers. The r-dependence is even non-analytical. Hence, no expansion in $r\xi$ with only integer powers holds.

From equations (3.5) and with $d_4 \ll d_2 \ll d_0$, we find that the leading terms in equations (3.5*b,c*) are $(d_0)^2 d_2 \sim r^{8/3}$ and $(d_0)^2 d_4 \sim r^{8/3}$. With $d_0 \sim r^{2/3}$ from equation (3.5*a*) this leads to $d_2 \sim d_4 \sim r^{4/3}$. The solutions of equations (3.5) for this non-analytic forcing describing shear flow are shown in figure 2. For $j = 0$ we recover the isotropic scaling of the structure function $d_0(r) \sim r^{2/3}$. However, for all higher amplitudes we obtain

$$d_j(r) \sim r^{4/3} \tag{4.6}$$

in a wide range of r. We have shown only the case where the isotropic forcing dominates, $e_0 \gg e_2 \gg e_4$. If the anisotropic contributions to the energy input increase, the r-range of scaling behaviour $d_0 \sim r^{2/3}$ and $d_j \sim r^{4/3}$ for all $j \geqslant 2$ is again shifted towards smaller r as in the case of analytic forcing.

As was argued by L'vov & Procaccia (1995*a,b*, 1996), in the exact resummation theory of Navier–Stokes turbulence no infrared (IR) divergence occurs if all j-factor scaling exponents ζ^j are bounded by 4/3. Otherwise IR-divergences cannot be excluded. Our results in the non-analytic case therefore exclude IR-divergences while in the previous analytic case they cannot be ruled out. Another possibility for controlling these IR divergences, in spite of second-order moment exponents being larger than 4/3, is by limiting the forcing to scales smaller than L, as recently shown in Kraichnan-model-type dynamics for a vector field, see Procaccia & Arad (2001).

5. Summary

Within a variable-scale mean-field theory of the Navier–Stokes equation we have derived the scaling exponents of the different j-amplitudes of the SO(3)-decomposition of the second-order structure function for weakly anisotropic turbulent flow. The limitation of this approach is its mean-field character. Clearly, intermittency effects cannot be captured, but we consider those to be small for second-order moments, to which the method is limited anyway. In the isotropic sector $j = 0$ we recover the classical scaling behaviour $\sim r^{2/3}$. The higher-order contributions, i.e. the anisotropic parts of the flow field, can be calculated order by order in Y_{jm}. They yield, for all j, the same mean-field scaling $r^{4/3}$ for a non-analytic forcing, whereas the scaling is $r^{j+2/3}$ for an analytic type of forcing. The non-analytic forcing might be more general, and therefore valid for a larger variety of anisotropic flows. Moreover, only the results for the non-analytic forcing are consistent with exisiting experimental measurements for the $j = 0$ and $j = 2$ amplitudes.

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